ON POINCARÉ TYPE INEQUALITIES

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ABSTRACT. Using estimates of the heat kernel we prove a Poincaré inequality for star-shape domains on a complete manifold. The method also gives a lower bound for the gap of the first two Neumann eigenvalues of a Schrödinger operator.

1. INTRODUCTION

Let M^m be an m-dimensional compact Riemannian manifold. In local coordinates $(x^1, x^2, \dots, x^m)$, the Riemannian metric is given by

$$ds^2 = \sum_{i,j=1}^{m} g_{ij} \, dx^i dx^j.$$

One defines the Laplace operator on M

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{m} \frac{\partial}{\partial x^i}(\sqrt{g}g^{ij}\frac{\partial}{\partial x^j}),$$

where $(g^{ij}) = (g_{ij})^{-1}$ and $g = det(g_{ij})$, and a Schrödinger operator by

$$-\Delta + q(x),$$

where $q(x)$ is a nontrivial C^1 function defined on M.

In this paper, we consider the Neumann eigenvalue problems on a manifold M with boundary ∂M for the Laplace operator

$$
\begin{aligned}
(1) \qquad -\Delta u &= \eta u \text{ in } M, \\
\frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial M,
\end{aligned}
$$

and the Schrödinger operator

$$
\begin{aligned}
(2) \qquad -\Delta u + qu &= \lambda u \text{ in } M, \\
\frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial M.
\end{aligned}
$$

When $\partial M = \emptyset$, we also consider a closed eigenvalue problem

$$(3) \qquad -\Delta u + qu = \lambda u \text{ in } M.$$

It is well known that the sets of eigenvalues $\{\eta_k\}$ and $\{\lambda_k\}$ are discrete and we can arrange them in a nondecreasing order,

$$0 = \eta_0 < \eta_1 \leq \eta_2 \leq \cdots \leq \eta_k \leq \cdots \to \infty$$

and

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots \to \infty.$$

In 1991, it was proved by Grigor'yan [G], and later independently by Saloff-Coste [S], that the parabolic Harnack inequality is equivalent to the volume doubling property and the Poincaré inequality. Therefore, it is interesting to find a lower bound for η_1 in terms of geometrical quantities. It is also natural to find a lower bound for the gap $\lambda_2 - \lambda_1$ in terms of the geometrical quantities of M and the potential function q. Both questions have been studied extensively by a long list of authors. We will simply refer the reader to Buser [B], Chavel [C], Cheeger [Ch], Chen [Cn1, Cn2], Li [L1], Li-Treibergs [LT], Li-Yau [LY1, LY2], Singer-Wong-Yau-Yau [SWYY], Yau [Y] and Zhong-Yang [ZY] for further references. Among all these results, Buser [B] first proved the Poincaré inequality for star-shaped domains in a Riemannian manifold by estimating the isoperimetric inequality by a geometric method. His estimate depends on the dimension, the lower bound of the Ricci curvature, the inner radius, and the outer radius of the domain. In particular, he did not assume any bound on the second fundamental form of the boundary. Later, Kusuoka and Stroock [KS] gave an argument using the heat equation to prove a weak Poincaré inequality for geodesic balls. The weak Poincaré inequality asserts that

$$C \inf_{a \in \mathbb{R}} \int_{B(R)} (u - a)^2 \leq \int_{B(2R)} |\nabla u|^2,$$

for any function $u \in H_{1,2}(B(2R))$ and for some constant $C > 0$ depending only on R, the dimension m, and the lower bound of the Ricci curvature on $B(2R)$. However, a covering argument in [J] asserts that the weak Poincaré inequality together with the volume doubling property will imply the Poincaré inequality for geodesic balls.

The first part of this paper is to show that one can use the heat kernel to prove the Poincaré inequality directly without using the covering argument. Also, this argument works on star-shaped domains. In particular, it give an analytic proof of Buser's theorem.

Theorem 1. *Let M be an m-dimensional manifold with boundary ∂M. Assume that M is geodesically star-shaped with respect to a point $p \in M$. Suppose that the Ricci curvature of M is bounded from below by $-(m-1)K$ for some constant $K \geq 0$. Let R be the radius of the largest geodesic ball centered at p contained in M, and R_0 be the radius of the smallest geodesic ball centered at p containing M. Then there exists a constant $C_1 > 0$ depending only on m, such that the first nonzero Neumann eigenvalue η_1 has a lower bound given by*

$$\eta_1 \geq \frac{R^m}{R_0^{m+2}} \exp(-C_1(1 + R_0 \sqrt{K})).$$

Corollary 1. *Let $B_p(R)$ be an m-dimensional geodesic ball centered at a point $p \in M$ with radius $R > 0$ such that $B_p(R) \cap \partial M = \emptyset$. Suppose that the Ricci curvature of $B_p(R)$ is bounded from below by $-(m-1)K$ for some constant $K \geq 0$.*

Then there exists a constant $C_2 > 0$ depending only on m, such that the first nonzero Neumann eigenvalue η_1 has a lower bound given by

$$\eta_1 \geq \frac{1}{R^2} \exp(-C_2(1 + R\sqrt{K})).$$

As an application, the lower bound on η_1 can be used to prove a lower bound for the first Dirichlet eigenvalue of proper subdomains of $B_p(R)$. In particular, let Ω be any proper subdomain in $B_p(R)$ such that $\bar{\Omega} \subset B_p(R)$. Let $\mu_1(\Omega)$ be the first Dirichlet eigenvalue satisfying

(4)
$$\begin{aligned} -\Delta u &= \mu_1 u &&\text{in } \Omega, \\ u &= 0 &&\text{on } \partial\Omega. \end{aligned}$$

(5)

By applying Theorem 1.4 of [G], we can estimate $\mu_1(\Omega)$ for any proper subdomain Ω. To state our theorems, it is convenient for us to give a precise definition of the volume doubling property.

Definition 1. *The manifold M is said to satisfy the volume doubling property with constant $C_v(R) > 0$ if for any point $p \in M$ and any positive number $r \leq R$, the ratio of the volumes of geodesic balls satisfies the estimate*

$$\frac{V(B_p(2r))}{V(B_p(r))} \leq C_v(R).$$

Note that the Bishop volume comparison theorem asserts that the constant $C_v(R)$ can be estimated from above by $R\sqrt{K}$.

Theorem 2. *Let $B_p(R)$ be as above. For any proper subdomain $\Omega \subset B_p(R)$, and for any function $u \in H_{1,2}(\Omega)$ which vanishes on $\partial\Omega$,*

$$\int_\Omega |\nabla u|^2 \geq \frac{\exp(-C_2(1 + R\sqrt{K}))}{R^2} \left\{ \frac{V(B_p(R))}{V(\Omega)} \right\}^{\alpha_1} \int_\Omega u^2,$$

for some constant $C_2 > 0$ depending only on m and some constant $\alpha_1 > 0$ depending only on m, and $R\sqrt{K}$.

Another advantage of this argument is that we can also estimate $\lambda_2 - \lambda_1$ from below for any Schrödinger operator $-\Delta + q$ defined on a geodesic ball. When the potential function q is assumed to be convex and if one imposes the Dirichlet condition on the boundary of a convex domain Ω in $\mathbb{R}^m$, Singer-Wong-Yau-Yau [SWYY] used the fact that the first eigenfunction is logarithmic concave to prove that the gap is bounded below by $\frac{\pi^2}{4d^2}$, where d is the diameter of Ω. Later, Zhong and Yang [ZY] improved the estimate to $\frac{\pi^2}{d^2}$. For the Neumann problem (2), we employ a method similar to Theorem 1 to get an estimate for the gap $\Gamma = \lambda_2 - \lambda_1$, of the Neumann problem (2). We also follow Grigor'yan's [G] argument to obtain a lower estimate for the gap of the Dirichlet problem in any proper subdomain of the geodesic ball.

Theorem 3. *Let M^m be as in Theorem 1 with $m \geq 3$. Let us denote $\omega(q) = \sup q - \inf q$ to be the oscillation of q over M. Then there exists a constant $\alpha_2 > 0$, depending only on m, $R_0\sqrt{K}$ and $\omega(q) R_0^2 + 1$, such that the gap of the first two Neumann eigenvalues $\Gamma = \lambda_2 - \lambda_1$ of (2) has a lower bound given by*

$$\Gamma \geq \frac{R^m}{R_0^{m+2}} \exp(-\alpha_2).$$

Corollary 2. *Let $B_p(R)$ be as in Corollary 1 with $m \geq 3$. Let us denote $\omega(q) = \sup q - \inf q$ to be the oscillation of q over $B_p(R)$. Then there exists a constant $\alpha_3 > 0$, depending only on m, $R\sqrt{K}$ and $\omega(q) R^2 + 1$, such that the gap of the first two Neumann eigenvalues $\Gamma = \lambda_2 - \lambda_1$ of (2) on $B_p(R)$ has a lower bound given by*

$$\Gamma \geq \frac{1}{R^2} \exp(-\alpha_3).$$

Using an argument similar to that of Theorem 2, we can also get the following lower bound for the gap of the first two Dirichlet eigenvalues of subdomains.

Theorem 4. *Let $B_p(R)$ be as in Corollary 1 with $m \geq 3$. For any proper subdomain $\Omega \subset B_p(R)$, if we let λ_1 and λ_2 be the first two eigenvalues of the problem*

$$\begin{aligned} -\Delta u + qu &= \lambda u \ in \ \Omega, \\ u &= 0 \ \ on \ \partial\Omega, \end{aligned}$$

then we have a lower estimate on the gap

$$\Gamma = \lambda_2 - \lambda_1 \geq \frac{\exp(-\alpha_4)}{R^2} \left\{ \frac{V(B_p(R))}{V(\Omega)} \right\}^{\alpha_5},$$

for some constants α_4, $\alpha_5 > 0$ depending only on m, $R\sqrt{K}$ and $\omega(q) R^2 + 1$.

Finally, we study the gap for a closed eigenvalue problem by using a heat kernel method.

Theorem 5. *Let M^m, $m \geq 3$, be a compact Riemannian manifold without boundary. Suppose that the Ricci curvature is bounded below by $-(m-1)K$, for some constant $K \geq 0$. If we let $\Gamma = \lambda_2 - \lambda_1$, where λ_1, λ_2 are the first two eigenvalues of the problem (3), then there exists a constant $C_3 > 0$ depending only on m such that*

$$\Gamma \geq \frac{1}{d^2} \exp\left[-C_3\left(1 + d[\sqrt{K} + \omega(q)]\right)\right],$$

where d is the diameter of M.

Remark. This improves the bound of the gap in [Cn1], where the bound on the gap depends on the lower bound of the sectional curvature, the bound of covariant derivatives of the curvatures and C^2 norm of the potential function q.

In Section 2, we first state two Harnack inequalities proved in [LY2] which will be used in the proofs of Theorem 1 and Theorem 3. Then we prove a Neumann Sobolev inequality on geodesic balls and a global Harnack inequality for the first eigenfunction of the Neumann Schrödinger problem (2). Also, we use a gradient estimate argument to prove a Harnack inequality for the first eigenfunction of the closed eigenvalue problem (3). In Section 3, we shall utilize the estimates obtained in Section 2 to prove our theorems.

2. Harnack inequalities

In this section, we recall two versions of the Harnack inequality for positive solutions of the heat equations from Li-Yau [LY2]. First we will state the local Harnack inequality given by Theorem 2.1 of [LY2].

Lemma 1. *Let $B_p(R) \subset M$ denote a geodesic ball of radius R centered at p with $B_p(R) \cap \partial M = \emptyset$. Assume that the Ricci curvature on $B_p(R)$ is bounded below by $-(m-1)K$. If $u(x,t)$ is a positive solution of the heat equation*

$$(6) \qquad (\Delta - \frac{\partial}{\partial t})u(x,t) = 0$$

on $B_p(R) \times (0,T]$, then for any $t > 0$ and $x, y \in B_p(\frac{R}{2})$, we have the inequality

$$u(x, 2t) \geq u(y,t)\, C_4 \, \exp\Big(- C_5\Big(\frac{R^2}{t} + Kt + \frac{t}{R^2}\Big)\Big),$$

for some constants C_4, $C_5 > 0$ depending only on m.

When the manifold is convex, Theorem 2.3 of [LY2] also asserts the following global Harnack inequality.

Lemma 2. *Let M be a compact manifold with Ricci curvature bounded below by $-(m-1)K$, for some constant $K \geq 0$. We assume the boundary of M is convex. Let $u(x,t)$ be a positive solution on $M \times (0,\infty)$ of the equation*

$$\Big(\Delta - \frac{\partial}{\partial t}\Big)u(x,t) = 0,$$

with Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0$$

on $\partial M \times [0,\infty)$. Then, for any $x, y \in M$, we have

$$u(x, 2t) \geq u(y,t)\, C_6 \, \exp\Big(- C_7(\frac{r^2(x,y)}{t} + Kt)\Big),$$

where $r(x,y)$ is the distance between x and y and C_6, $C_7 > 0$ are constants depending only on m.

To prove Theorem 3, we need to use a global version of Moser's iteration scheme in which a Neumann Sobolev inequality on geodesic balls will be needed. We shall follow the covering argument used by Jerison in [J]. We first prove the following local version of the Sobolev inequality.

Lemma 3. *Let M be a complete manifold with Ricci curvature bounded from below by $-(m-1)K$, for some constant $K \geq 0$. Let $y \in M$ and $R > 0$ such that $B_y(2R) \subset M$. Then, for any function $f \in H_{1,2}(B_p(2R))$, there exist constants C_8, $C_9 > 0$ depending on m such that*

$$(7) \qquad C_s(R) \int_{B_y(2R)} |\nabla f|^2 \geq \min_{a \in \mathbb{R}} \Big(\int_{B_y(R)} |f - a|^{\frac{2m}{m-2}} \Big)^{\frac{m-2}{m}},$$

where $C_s(R) = C_8 \, R^2 \, V(B_p(R))^{\frac{-2}{m}} \, \exp(C_9 K R^2)$.

Proof. Let $\phi(r)$ be a cut-off function defined on the interval $[0, \infty)$, such that

$$(8) \qquad \phi(r) = \begin{cases} 1 & \text{for } 0 \le r \le 1, \\ 0 & \text{for } 2 \le r, \end{cases}$$

and $0 \ge \phi'(r) \ge -2$. Let $f_0 = V(B_y(2R))^{-1} \int_{B_y(2R)} f$ and $\varphi(x) = \phi(\frac{r(x)}{R})$, where $r(x)$ is the distance from y to x. The Dirichlet Sobolev inequality for geodesic balls implies that there exists constants C_{10}, $C_{11} > 0$, and

$$s(R) = C_{10} \, R^2 \, V(B_y(R))^{\frac{-2}{m}} \, \exp(C_{11} K R^2),$$

such that

$$
\begin{aligned}
&\min_{a \in \mathbb{R}} \left(\int_{B_y(R)} |f - a|^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} \\
&\le \left(\int_{B_y(R)} |f - f_0|^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} \\
&\le \left(\int_{B_y(2R)} |\varphi (f - f_0)|^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} \\
&\le s(2R) \int_{B_y(2R)} |\nabla[\varphi (f - f_0)]|^2 \\
&\le 2 \, s(2R) \left(\int_{B_y(2R)} \varphi^2 \, |\nabla f|^2 + |\nabla \varphi|^2 (f - f_0)^2 \right) \\
&\le 2 \, s(2R) \left(\int_{B_y(2R)} \varphi^2 \, |\nabla f|^2 + \frac{4}{r^2} \int_{B_y(2R)} (f - f_0)^2 \right) \\
&\le 2 \, s(2R) \left(\int_{B_y(2R)} |\nabla f|^2 + 16 \exp\left(C_2(1 + 2R\sqrt{K}) \right) \int_{B_y(2R)} |\nabla f|^2 \right) \\
&\le C_s(R) \int_{B_y(2R)} |\nabla f|^2,
\end{aligned}
$$

(9)

where we have used the result of Corollary 1 in the second last inequality. This completes the proof of the lemma. $\qquad\square$

We shall modify some results of Whitney decomposition from [J]. Since the Ricci curvature is bounded from below, the inequalities (28), (29) in the next section imply that the volume doubling property holds on M. For a ball $B = B_y(r)$, denote $B' = B_y(2r)$, $B'' = B_y(4r)$ and $B^* = B_y(10r)$. We shall also denote the radius of B by $\rho(B)$.

Lemma 4. (Whitney decomposition) *Let M be as in Theorem 1. There is a pairwise disjoint family of balls $\mathcal{F}$ and a constant α_6 depending only on the volume doubling property $C_v(R_0)$ such that*

(a) $M = \bigcup_{B \in \mathcal{F}} B'$
(b) $B \in \mathcal{F}$ *implies that* $10^2 \rho(B) \le r(B, \partial M) \le 10^3 \rho(B)$, *where $r(B, \partial M)$ is the distance from B to ∂M.*
(c) $\#\{B \in \mathcal{F} : x \in M, x \in B^*\} \le \alpha_6$, *where $\#S$ is the number of elements in S.*

For $B \in \mathcal{F}$, define γ_B as an admissible geodesic from the center of B to p of length $\le R_0$. This path may not be unique, but will be fixed throughout the argument. Denote $\mathcal{F}(B) = \{A \in \mathcal{F} : A' \cap \gamma_B \ne \emptyset\}$.

Lemma 5. *Let M be as in Theorem 1 and B belong to $\mathcal{F}$.*

(a) *There are no elements of $\mathcal{F}(B)$ of radius less than $\dfrac{\rho(B)R}{100R_0} \exp(-C(m)\sqrt{K}R_0)$.*

(b) *For any r, $\#\{A \in \mathcal{F}(B) : r \leq \rho(A) \leq 2r\} \leq \alpha_7$, a constant depending on the volume doubling constant $C_v(R_0)$.*

Proof. (a) Suppose that $A \in \mathcal{F}(B)$ and $\eta \in A' \cap \gamma_B$ then $(10^3+2)\rho(A) \geq r(\eta, \partial M) \geq 98\rho(A)$. Let $b \in \overline{B(\eta, r)} \cap \partial M$, where $r = r(\eta, \partial M)$. Also, we let η_1 denote the center of B and $b_1 \in \overline{B(\eta_1, r(B, \partial B))} \cap \partial M$. Since M is geodesically star-shaped, the point b will lie in a "sector" bounded between the geodesic γ_B and the geodesic joining p to b. Furthermore, since the Ricci curvature is bounded from below by $-K$, this implies that $\overline{B(\eta_1, s)} \cap \partial M \neq \emptyset$ for $s = \dfrac{r\,R_0}{R} \exp(C(m)\sqrt{K}R_0)$ and, hence, $\rho(A) \geq \frac{\rho(B)\,R}{100\,R_0} \exp(-C(m)\sqrt{K}R_0)$.

(b) An argument nearly identical to the one in part (a) shows that if $A \in \mathcal{F}(B)$ and $r \leq \rho(A) \leq 2r$, then $A \subset B(\eta_1, 10^5 s)$. Next, since $\mathcal{F}$ is a disjoint family

$$\sum_{\substack{A \in \mathcal{F}(B) \\ r \leq \rho(A) \leq 2r}} V(A) \leq V(B(\eta_1, 10^5 s)).$$

On the other hand since $\rho(A) \geq r$, the volume doubling condition implies that there is an α_7 depending on $R_0\sqrt{K}$ such that $V(B(\eta_1, 10^5 s)) \leq \alpha_7 V(A)$. In all, $\#\{A \in \mathcal{F}(B) : r \leq \rho(A) \leq 2r\} \leq \alpha_7$. □

It follows immediately from this lemma and volume comparison theorem for star-shaped domain in [CGT, Remark 4.1] that

Corollary 3. *Let M be as in Theorem 1. There exits a constant α_8 depending on $C_v(R_0)$ such that*

$$\#\mathcal{F}(B) \leq \alpha_8 \log\left(\frac{R_0}{\rho(B)}\right).$$

Lemma 6. *Let M be as in Theorem 1. There exists a constant α_9 depending on $C_v(R_0)$ and $\epsilon > 0$ such that for any $A \in \mathcal{F}$ and any $r > 0$,*

$$\sum_{\substack{B \in A(\mathcal{F}) \\ r \leq \rho(B) \leq 2r}} V(B) \leq \alpha_9 \left(\frac{\rho(B)}{\rho(A)}\right)^{\epsilon} V(A),$$

where $V(A)$ denotes the volume of the ball A and $A(\mathcal{F}) = \{B \in \mathcal{F} : A \in \mathcal{F}(B)\}$.

Also, we need the following lemma.

Lemma 7. *Let M be as in Theorem 1. For any positive constants l_1 and l_2 there is $l_3 = l_3(l_1, l_2)$ such that for every pair of balls B_1, $B_2 \subset M$ satisfying*

$$l_1 V(B_1) \leq V(B_1 \cap B_2) \leq l_2 V(B_2)$$

and a function f such that

$$\left(\int_{B_j} |f - f_{B_j}|^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq \theta$$

for $j = 1, 2$, we have

$$\left(\int_{B_1 \cup B_2} |f_{B_1} - f_{B_2}|^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} \leq l_3 \theta.$$

In the above inequalities, the constants f_{B_j} for $j = 1, 2$, are defined by the formula

$$\int_{B_j} |f - f_{B_j}|^{2m/m-2} = \inf_{a \in \mathbf{R}} \int_{B_j} |f - a|^{2m/m-2}.$$

Proof. There exists a constant l_4 such that

$$\left(\int_{B_1 \cap B_2} |f_{B_1} - f_{B_2}|^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}}$$

$$\leq l_4 \left(\int_{B_1 \cap B_2} |f_{B_1} - f|^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} + l_4 \left(\int_{B_1 \cap B_2} |f - f_{B_2}|^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}}$$

$$\leq 2l_4 \theta.$$

Hence, there exists a constant l_5 such that

$$\left(\int_{B_1 \cup B_2} |f - f_{B_1}|^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}}$$

$$\leq l_5 \Bigg\{ \left(\int_{B_1} |f - f_{B_1}|^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}}$$

$$+ \left(\int_{B_2} |f - f_{B_2}|^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} \left(\int_{B_2} |f_{B_1} - f_{B_2}|^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} \Bigg\}$$

$$\leq 2l_5 \theta + 2l_4 \theta l_5 \frac{V(B_2)}{V(B_1 \cap B_2)}.$$

This completes the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\square$

We shall apply the above results to prove the following weaker version of Sobolev inequality.

Lemma 8. *Let M be as in Theorem 1 and $p \in M$. For any function $f \in H_{1,2}(M)$, there exist constants $C_{12}, C_{13} > 0$ depending on m such that*

$$(10) \qquad\qquad C_s(R_0) \int_M |\nabla f|^2 \geq \min_{a \in R} \left(\int_M |f - a|^{\frac{2m}{m-1}} \right)^{\frac{m-1}{m}},$$

where $C_s(R_0) = C_{12} R_0^2 V(B_p(R_0))^{\frac{-2}{m}} \exp(C_{13} K R_0^2)$.

Proof. Let $\mathcal{F}$ be the Whitney family of Lemma 4. Let $f \in C^\infty(\bar{M})$. Choose $B_0 \in \mathcal{F}$ such that $p \in B_0'$. Order the elements of $\mathcal{F}(B) = \{A_1, A_2, \dots, A_l\}$ so that $A_1 = B$, $A_l = B_0$, and $A_k' \cap A_{k+1}' \neq \emptyset$ for all k. Note that A_k'' is contained in the ball with the same center as A_k^* and half the radius. Lemma 3 implies that

$$\left(\int_{A_k''} |f - f_{A_k''}|^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} \leq C_s(2\rho(A_k)) \int_{A_k^*} |\nabla f|^2.$$

Moreover, the volume doubling property implies that A_k and A_{k+1} have comparable radii and volume and $A_k'' \cap A_{k+1}''$ contains a ball of comparable volume to these. Thus Lemma 7 implies that there is a constant C_{14} such that

$$\left(\int_{A_k'' \cup A_{k+1}''} |f_{A_k''} - f_{A_{k+1}''}|^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} \leq C_{14} C_s(2\rho(A_k)) \int_{A_k^* \cup A_{k+1}^*} |\nabla f|^2.$$

Therefore, there exists constants C_{15}, $C_{16} > 0$, such that

$$\left(\int_{B'} |f - f_{B_0''}|^{\frac{2m}{m-1}}\right)^{\frac{m-1}{m}}$$

$$\leq \left(\int_{B'} |f - f_{B_0''}|^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} V(B')^{\frac{1}{m}}$$

$$= \left(\int_{B'} |f - f_{B''} + \sum_{k=1}^{l-1}(f_{A_k''} - f_{A_{k+1}''})|^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} V(B')^{\frac{1}{m}}$$

$$\leq C_{15}\, l \left(\int_{B'} |f - f_{B''}|^{\frac{2m}{m-2}} + \sum_{k=1}^{l-1} |f_{A_k''} - f_{A_{k+1}''}|^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} V(B')^{\frac{1}{m}}$$

$$\leq C_{16}\, l \sum_{k=1}^{l} C_s(2\rho(A_k)) \frac{V(B)^{\frac{m+1}{m}}}{V(A_k)} \int_{A_k^*} |\nabla f|^2$$

$$= C_{16}\, \#\mathcal{F}(B) \sum_{A_k \in \mathcal{F}(B)} C_s(2\rho(A_k)) \frac{V(B)^{\frac{m+1}{m}}}{V(A_k)} \int_{A_k^*} |\nabla f|^2.$$

Summing over all $B \in \mathcal{F}$ and using Lemma 4(a), we obtain

$$\left(\int_{M} |f - f_{B_0''}|^{\frac{2m}{m-1}}\right)^{\frac{m-1}{m}}$$

$$\leq \sum_{B \in \mathcal{F}} \left(\int_{B'} |f - f_{B_0''}|^{\frac{2m}{m-1}}\right)^{\frac{m-1}{m}}$$

$$\leq C_{16} \sum_{B \in \mathcal{F}} \sum_{A \in \mathcal{F}(B)} \#\mathcal{F}(B) C_s(2\rho(A)) \frac{V(B)^{\frac{m+1}{m}}}{V(A)} \int_{A^*} |\nabla f|^2$$

$$\leq C_{16} \sum_{A \in \mathcal{F}} \sum_{B \in A(\mathcal{F})} \#\mathcal{F}(B) C_s(2\rho(A)) \frac{V(B)^{\frac{m+1}{m}}}{V(A)} \int_{A^*} |\nabla f|^2.$$

From Corollary 3 and Lemma 6 there exists a constant C_{17}, $C_{18} > 0$ such that

$$\sum_{B \in A(\mathcal{F})} \#\mathcal{F}(B)\, V(B)^{\frac{m+1}{m}} = \sum_{k=0}^{\infty} \sum_{\substack{B \in A(\mathcal{F}) \\ 2^{-k}\rho(A) < \frac{\rho(B)}{100} < 2^{-k+1}\rho(A)}} \#\mathcal{F}(B)\, V(B)^{\frac{m+1}{m}}$$

$$\leq C_{17} \sum_{k=0}^{\infty} \left(k + \log \frac{R_0}{\rho(A)}\right) 2^{-\epsilon k}\, V(A)^{\frac{m+1}{m}}$$

$$\leq C_{18}\, \log\left(\frac{R_0}{\rho(A)}\right) V(A)^{\frac{m+1}{m}}.$$

Because $\rho(A) < \frac{R_0}{10}$, there exists a constant C_{19} such that

$$C_s(2\rho(A))\, V(A)^{\frac{1}{m}}\, \log\left(\frac{R_0}{\rho(A)}\right) \leq C_{19} C_s(R_0).$$

Combining the above estimates together, there exists a constant C_{20} such that

$$\left(\int_M |f - f_{B_0''}|^{\frac{2m}{m-1}} \right)^{\frac{m-1}{m}} \leq C_{16} \sum_{A \in \mathcal{F}} C_{19} C_s(R_0) \int_{A^*} |\nabla f|^2$$

$$\leq C_{20} C_s(R_0) \int_M |\nabla f|^2$$

by Lemma 4(c). This completes the proof of our lemma. $\qquad\square$

Using the above Sobolev inequality along with Moser's iteration scheme, we obtain the following global version of Harnack inequality for the first Neumann eigenfunction u_1 of the equation (2).

Corollary 4. *Let M be as in Theorem 1 and u_1 be the first Neumann eigenfunction of the equation (2) on M. Then, for any $x, y \in M$, we have*

$$u_1(x) \geq C_h(M) \, u_1(y),$$

where $1 > C_h(M)$ depends on m, $C_s(R_0)$, η_1, C_v, and $\omega(q) R_0^2 + 1$.

For the sake of completeness, we shall include a proof here.

Proof. By modifying arguments in [L1], it suffices to prove the following two lemmas. $\qquad\square$

Lemma 9. *Let M, u be as in the theorem. For any $k > 0$, there exists a positive constant*

$$\alpha_{11} = \left[\left(\frac{k\omega(q)}{\alpha_{10}} \right)^{\frac{m}{2k}} + 2^{\frac{m}{2k}} V(M)^{\frac{-1}{k}} \right] \left(\frac{m}{m-1} \right)^{\frac{1}{k} \sum_{j=0}^{\infty} j(\frac{m-1}{m})^j},$$

where $\alpha_{10} = \dfrac{D(m)}{C_s(R_0)}$ for some constant $D(m)$ depending only on m, such that

$$\|u\|_\infty \leq \alpha_{12} \| \, u \|_k,$$

where $\alpha_{12} = \max\{\alpha_{11}, \left(\alpha_{11}^2 V(M) \right)^{\frac{1}{k}}\}$.

Proof. For any constant $a \geq 1$ and $\mu = \dfrac{m}{m-1}$. the assumption of u implies that

$$\omega(q) \int_M u^{2a} \geq - \int_M u^{2a-1} \Delta u.$$

Integrating by parts and using the boundary condition, the right-hand side yields

$$- \int_M u^{2a-1} \Delta u = (2a-1) \int_M u^{2a-2} |\nabla u|^2$$

$$\geq a \int_M u^{2a-2} |\nabla u|^2.$$

Using a similar argument as in [L2], the Sobolev inequality (10) implies that for all $f \in H_{1,2}(M)$, there exists constant $D(m)$ and $\alpha_{10} = \dfrac{D(m)}{C_s(R_0)}$ such that

$$(11) \qquad \int_M |\nabla f|^2 \geq \alpha_{10} \left[\left(\int_M |f|^{\frac{2m}{m-1}} \right)^{\frac{m-1}{m}} - V(M)^{-\frac{1}{m}} \int_M f^2 \right].$$

This implies that

$$a^2 \int_M u^{2a-2} |\nabla u|^2 = \int_M |\nabla u^a|^2$$
$$\geq \alpha_{10} \left[\left(\int_M u^{2a\mu} \right)^{\frac{1}{\mu}} - V(M)^{-\frac{1}{m}} \int_M u^{2a} \right].$$

We get

$$\left(\int_M u^{2a\mu} \right)^{\frac{1}{\mu}} \leq \left(\frac{\omega(q)a}{\alpha_{10}} + V(M)^{-\frac{1}{m}} \right) \int_M u^{2a},$$

which is equivalent to

(12) $$\left(\frac{\omega(q)a}{\alpha_{10}} + V(M)^{-\frac{1}{m}} \right)^{\frac{1}{2a}} \|u\|_{2a} \geq \|u\|_{2a\mu}.$$

Let us choose a sequence of a_i such that

$$a_0 = \frac{k}{2},\, a_1 = \frac{k\mu}{2},\cdots,\, a_i = \frac{k\mu^i}{2},\, \cdots.$$

Applying (12) to $a = a_i$ and iterating the inequality, we conclude that

$$\|u\|_{2a_{i+1}} \leq \prod_{j=0}^{i} \left(\frac{a_j\omega(q)}{\alpha_{10}} + V(M)^{-\frac{1}{m}} \right)^{\frac{1}{2a_j}} \|u\|_k.$$

On the other hand, we have

$$\lim_{i\to\infty} \| u\|_{2a_{i+1}} = \|u\|_\infty.$$

Therefore, letting $i \to \infty$, we conclude that

$$\|u\|_\infty \leq \prod_{j=0}^{\infty} \left(\frac{k\mu^j\omega(q)}{2\alpha_{10}} + V(M)^{-\frac{1}{m}} \right)^{\frac{1}{k\mu^j}} \|u\|_k.$$

The product can be estimated by using the fact that

$$\prod_{j=0}^{\infty} B^{\mu^{-j}} = B^{\frac{\mu}{\mu-1}}$$

and the fact that $\displaystyle\sum_{j=0}^{\infty} j\mu^{-j}$ is finite. Hence we have

$$\prod_{j=0}^{\infty} \left(\frac{k\mu^j\omega(q)}{2\alpha_{10}} + V(M)^{-\frac{1}{m}} \right)^{\frac{1}{k\mu^j}}$$
$$\leq \prod_{j=0}^{\infty} \left[\left(\frac{k\omega(q)}{2\alpha_{10}} + V(M)^{-\frac{1}{m}} \right)\mu^j \right]^{\frac{1}{k\mu^j}}$$
$$\leq \left(\frac{k\omega(q)}{2\alpha_{10}} + V(M)^{-\frac{1}{m}} \right)^{\frac{\mu}{k(\mu-1)}} \mu^{\frac{1}{k}\sum_{j=0}^{\infty} j\mu^{-j}}$$
$$\leq \left[\left(\frac{k\omega(q)}{\alpha_{10}} \right)^{\frac{m}{2k}} + 2^{\frac{m}{2k}} V(M)^{\frac{-1}{k}} \right] \left(\frac{m}{m-1} \right)^{\frac{1}{k}\sum_{j=0}^{\infty} j(\frac{m-1}{m})^j}$$
$$= \alpha_{11}.$$

We have proved that

$$\|u\|_\infty \leq \alpha_{11} V(M)^{\frac{1}{k}} \| u\|_k$$

for $a \geq 1$, or equivalently, for $k \geq 2$. For those values $k < 2$, we begin with the case $k = 2$. In that case , the inequality takes the form

$$
\begin{aligned}
\|u\|_\infty &\leq \alpha_{11} \, V(M)^{\frac{1}{2}} \| u\|_2 \\
&\leq \alpha_{11} \|u\|_k^{\frac{k}{2}} \|u\|_\infty^{1-\frac{k}{2}}.
\end{aligned}
$$

Iterating the inequality yields

$$
(13) \qquad \|u\|_\infty \leq \|u\|_\infty^{(1-\frac{k}{2})^i} \prod_{j=1}^{i} \left[\alpha_{11} \|u\|_k^{\frac{k}{2}} \right]^{(1-\frac{k}{2})^{j-1}}.
$$

Letting $i \to \infty$, the term

$$
\|u\|_\infty^{(1-\frac{k}{2})^i} \to 1,
$$

and

$$
\prod_{j=1}^{\infty} \|u\|_k^{\frac{k}{2}(1-\frac{k}{2})^{j-1}} = \|u\|_k.
$$

Hence, (13) implies that

$$
\|u\|_\infty \leq \left(\alpha_{11}^2 V(M) \right)^{\frac{1}{k}} \| u\|_k.
$$

This proves our lemma. □

Lemma 10. *Let M, u be as in the theorem. For $k > 0$ sufficiently small, there exists constant α_{17} depending on m, k, $V(M)$, η_1, $C_s(R_0)$ and $\omega(q)$ such that*

$$
\| u\|_k \leq \frac{1}{\alpha_{12}^2 \, \alpha_{17}^2} \inf_M u = C_h \inf_M u.
$$

Proof. The function u^{-1} satisfies that

$$
\Delta u^{-1} = -u^{-2}\Delta u + 2u^{-3}|\nabla u|^2 \geq -\omega(q) \, u^{-1}.
$$

By applying Lemma 9 to u^{-1}, we have

$$
\left(\inf_M u \right)^{-1} = \sup_M u^{-1} \leq \alpha_{12} \| u^{-1}\|_k.
$$

Clearly, the lemma follows if we can estimate the product

$$
\| u^{-1}\|_k \cdot \| u\|_k
$$

from above for some value of $k > 0$. To achieve this, let us consider the function

$$
v = \beta + \log u
$$

where $\beta = \dfrac{-1}{V(M)} \displaystyle\int_M \log u$. The function v satisfies

$$
\Delta v = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} \leq \omega(q) - |\nabla v|^2,
$$

hence

$$
(14) \qquad |\nabla v|^2 \leq \omega(q) - \Delta v.
$$

Integrating and using the boundary condition of u, we deduce that

$$
(15) \qquad \int_M |\nabla v|^2 \leq \omega(q) \, V(M).
$$

However, the choice of β and Theorem 1 implies that

$$\eta_1 \int_M v^2 \leq \int_M |\nabla v|^2.$$

Hence, we have

(16)
$$\int_M v^2 \leq \frac{\omega(q)\,V(M)}{\eta_1}.$$

Applying the Schwarz inequality, we also have

(17)
$$\int_M |v| \leq \left(\frac{\omega(q)}{\eta_1}\right)^{\frac{1}{2}} V(M).$$

Multiplying $|v|^{2a-2}$ to (14) for $a \geq 2$, and integrating by parts yields

(18)
$$\int_M |v|^{2a-2} |\nabla v|^2 \omega(q) \int_M |v|^{2a-2} - \int_M |v|^{2a-2}\,\Delta v$$
$$\leq \omega(q) \int_M |v|^{2a-2} + (2a-2) \int_M |v|^{2a-3} |\nabla v|^2.$$

Using (15) and the inequality

$$(2a-2) \int_M |v|^{2a-3} |\nabla v|^2 \leq \frac{1}{4} \int_M |v|^{2a-2} |\nabla v|^2 + (8a-12)^{2a-3} \int_M |\nabla v|^2,$$

(18) becomes

(19)
$$\int_M |v|^{2a-2} |\nabla v|^2 \leq 2\omega(q) \int_M |v|^{2a-2} + 2(8a-12)^{2a-3} \int_M |\nabla v|^2$$
$$\leq 2\omega(q) \left(\int_M |v|^{2a-2} + V(M)(8a-12)^{2a-3} \right).$$

By setting $a = 2$ and combining with (16), we have

$$\int_M v^2 |\nabla v|^2 \leq \alpha_{13}$$

for some constant $\alpha_{13} = \dfrac{2\,\omega(q)^2\,V(M)}{\eta_1} + 8\,\omega(q)V(M)$. On the other hand,

$$
\begin{aligned}
\int_M v^2 |\nabla v|^2 &= \frac{1}{4} \int_M |\nabla(sgn(v)v^2|^2 \\
&\geq \frac{\eta_1}{4} \int_M \left(sgn(v)v^2 - \frac{1}{V(M)} \int_M sgn(v)v^2 \right)^2 \\
&\geq \frac{\eta_1}{4} \left\{ \int_M |v|^4 - \frac{1}{V(M)} \left(\int_M sgn(v)v^2 \right)^2 \right\} \\
&\geq \frac{\eta_1}{4} \left\{ \int_M |v|^4 - \frac{1}{V(M)} \left(\int_M v^2 \right)^2 \right\}.
\end{aligned}
$$

Hence combining with (19), we have

(20)
$$\int_M |v|^4 \leq \alpha_{14}$$

for some constant $\alpha_{14} = \dfrac{9\,\omega(q)^2\,V(M)}{\eta_1^2} + \dfrac{32\,\omega(q)\,V(M)}{\eta_1}$. Using Schwarz inequality, we also deduce that

$$(21) \qquad \int_M |v|^3 \le \alpha_{14}^{\frac{3}{4}}\, V(M)^{\frac{1}{4}}.$$

For general $a \ge 2$, note that

$$|\nabla(|v|^a)|^2 = a^2 |v|^{2a-2}\,|\nabla v|^2.$$

Combining this with (19), we conclude that

$$\int_M |\nabla(|v|^a)|^2 \;\le\; 2a^2\,\omega(q)\Big(\int_M |v|^{2a-2} + V(M)(8a-12)^{2a-3}\Big).$$

Using the inequality

$$|v|^{2a-2} \le |v|^{2a} + 1,$$

we have

$$\int_M |\nabla(|v|^a)|^2 \;\le\; \omega(q)\Big(2a^2 \int_M |v|^{2a} + V(M)(8a-12)^{2a-1}\Big).$$

Hence, applying Sobolev inequality (11),

$$\alpha_{10}\left[\Big(\int_M (|v|^a)^{2\mu}\Big)^{\frac{1}{\mu}} - V(M)^{-\frac{1}{m}} \int_M (|v|^a)^2\right] \;\le\; \int_M |\nabla(|v|^a)|^2,$$

we conclude that

$$\Big(\int_M |v|^{2a\mu}\Big)^{\frac{1}{2a\mu}}$$
$$\le\; \Big(\frac{2\,a^2\,\omega(q)}{\alpha_{10}} + V(M)^{-\frac{1}{m}}\Big)^{\frac{1}{2a}} \Big(\int_M |v|^{2a}\Big)^{\frac{1}{2a}} + \Big(\frac{\omega(q)\,V(M)}{\alpha_{10}}\Big)^{\frac{1}{2a}} (8a).$$

Consider the sequence

$$a_0 = 2,\; a_1 = 2\mu,\; \cdots,\; a_i = 2\mu^i,\; \cdots.$$

Applying the inequality to a_i, we have

$$\Big(\int_M |v|^{4\mu^{i+1}}\Big)^{\frac{1}{4\mu^{i+1}}}$$
$$\le\; \Big(\frac{2\,\omega(q)}{\alpha_{10}} + V^{-\frac{1}{m}}\Big)^{\frac{1}{4\mu^i}} (2\mu^i)^{\frac{1}{2\mu^i}} \Big(\int_M |v|^{4\mu^i}\Big)^{\frac{1}{4\mu^i}} + \Big(\frac{\omega(q)\,V(M)}{\alpha_{10}}\Big)^{\frac{1}{4\mu^i}} (16\mu^i).$$

Iterating this by running $i = 0, \cdots, l$ gives

$$\|v\|_{4\mu^{l+1}} \le \prod_{i=0}^{l} \Big(\frac{2\,\omega(q)}{\alpha_{10}} + V^{-\frac{1}{m}}\Big)^{\frac{1}{4\mu^i}} (2\mu^i)^{\frac{1}{2\mu^i}} \|v\|_4 + \Big(\frac{\omega(q)\,V(M)}{\alpha_{10}}\Big)^{\frac{1}{4\mu^l}} (16\mu^l)$$
$$+ \sum_{i=0}^{l-1} \Big(\frac{\omega(q)\,V(M)}{\alpha_{10}}\Big)^{\frac{1}{4\mu^i}} (16\mu^i) \prod_{j=i+1}^{l} \Big(\frac{2\,\omega(q)}{\alpha_{10}} + V^{-\frac{1}{m}}\Big)^{\frac{1}{4\mu^j}} (2\mu^j)^{\frac{1}{2\mu^j}}$$

Using the equality $\sum\limits_{i=0}^{\infty} \mu^{-i} = \dfrac{m}{2}$, and the fact that $\sum\limits_{i=0}^{\infty} i\mu^{-i}$ is finite, we conclude that

(22)
$$\| v\|_{4\mu^{l+1}} \leq \left\{ 16\Big(\frac{4\,\omega(q)}{\alpha_{10}}\Big)^{\frac{m}{4}} + 2^{\frac{m}{4}}\, V^{\frac{-1}{4}} \right\} \Big(\frac{m}{m-1}\Big)^{\frac{1}{k}\sum_0^\infty i(\frac{m-1}{m})^i} \Big(\| v\|_4 + \sum_{i=0}^{l} \mu^i \Big)$$

$$\leq \alpha_{15}\Big(\| v\|_4 + 4\mu^l \Big).$$

For each integer $j \geq 4$, let l be such that $4\mu^l < j < 4\mu^{l+1}$. Using the Hölder inequality and the estimate (22), we get

$$\fint_M |v|^j \;\leq\; \Big(\fint_M |v|^{4\mu^{l+1}} \Big)^{\frac{j}{4\mu^{l+1}}}$$

$$\leq\; \alpha_{15}^j \Big(\| v\|_4 + j \Big)^j.$$

Combining this with (16), (17), (20) and (21), we have

$$\fint_M e^{k|v|} \;=\; \sum_{j=0}^{\infty} \frac{k^j}{j!} \fint_M |v|^j$$

$$\leq\; \Big(V(M) + \sum_{j=1}^{4} \frac{k^j}{j!} \, (\alpha_{14}\, V(M))^{1-\frac{j}{4}} \Big) + \sum_{j=5}^{\infty} \frac{\big[\alpha_{15}(\alpha_{14}+1)kj\big]^j}{j!}$$

$$\leq\; \alpha_{16} + \sum_{j=5}^{\infty} \frac{\big[\alpha_{15}\,(\alpha_{14}+1)\,k\,j\big]^j}{j!}$$

However, using Stirling's inequality $j^j < j!\, e^j$, we conclude that

$$\fint_M e^{k|v|} \leq \alpha_{16} + \sum_{j=5}^{\infty} \big[\alpha_{15}\,(\alpha_{14}+1)\,k\,e\big]^j.$$

Therefore, by choosing $k < \big[\alpha_{15}\,(\alpha_{14}+1)\,e\big]^{-1}$, the infinite series converges and we obtain the estimate

$$\fint_M e^{k|v|} \leq \alpha_{17}$$

where $\alpha_{17} = \alpha_{16} + \sum\limits_{j=5}^{\infty} \big[\alpha_{15}\,(\alpha_{14}+1)\,k\,e\big]^j$. Let us now observe that

$$e^{k\beta}\, u^k = e^{kv} \leq e^{k|v|}$$

and

$$e^{-k\beta}u^{-k} = e^{-kv} \leq e^{k|v|}$$

imply that

$$\| u^{-1}\|_k \cdot \| u\|_k \leq \Big(\fint_M e^{k|v|} \Big)^2.$$

This implies that

$$
\begin{aligned}
\inf_M u \;\; &\geq\;\; \frac{1}{\alpha_{12}\,\|\!\|\,u^{-1}\|_k} \\
&\geq\;\; \frac{1}{\alpha_{12}\,\alpha_{17}^2}\,\|\!\|\,u\|_k \\
&\geq\;\; \frac{1}{\alpha_{12}^2\,\alpha_{17}^2}\,\sup_M u.
\end{aligned}
$$

$\square$

To prove Theorem 5, we also need to obtain a Harnack inequality for the first eigenfunction. We use the following gradient estimate of the first eigenfunction to obtain a Harnack inequality.

Lemma 11. *Let M be a compact manifold without boundary. Suppose that the Ricci curvature of M is bounded below by $-(m-1)K$, for some nonnegative constant K. Let u_1 be a first eigenfunction of the equation (3) on M. Then, for any $x, y \in M$, we have*

$$|\nabla \log u_1(x)|^2 \leq \alpha_{18},$$

for some constant α_{18} defined by

$$\alpha_{18} = m\,(K+2) + \sqrt{2\,m\,|\nabla q|^2 + \frac{1}{2}\omega(q)^2}.$$

Proof. Let $f = \log u_1$ and $g = |\nabla f|^2$. Then, we have

$$\Delta f = (q - \lambda_1) - |\nabla f|^2.$$

Since M is compact, there exists a point $x_0 \in M$ such that $g(x_0) = \max_{x \in M} g(x)$. Therefore, at x_0,

$$\nabla g(x_0) \;\;=\;\; 0,$$

and

$$\Delta g(x_0) \;\;\leq\;\; 0.$$

Direct computations give us, at x_0,

$$
\begin{aligned}
0 \;\;&\geq\;\; \frac{1}{m}\,(\Delta f)^2 + \langle \nabla f, \nabla g \rangle - K|\nabla f|^2 \\
&\geq\;\; \frac{1}{m}\left(|\nabla f|^4 - \frac{1}{2}\omega(q)^2\right) - 2|\nabla q|^2 - 2|\nabla f|^2 - K|\nabla f|^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
g(x) \;\;&\leq\;\; g(x_0) \\
&\leq\;\; m\,(K+2) + \sqrt{2\,m\,|\nabla q|^2 + \frac{1}{2}\omega(q)^2}.
\end{aligned}
$$

It is easy to see that a Harnack inequality follows from the above gradient estimate.

$\square$

Corollary 5. *Let M be a compact manifold without boundary. Suppose that the Ricci curvature of M is bounded below by $-(m-1)K$, for some nonnegative constant K. Let u_1 be a first eigenfunction of the equation (3) on M. Then, for any $x, y \in M$, we have*

$$u_1(x) \geq e^{-\sqrt{\alpha_{18}}\,d}u_1(y).$$

3. Proofs

In this section, we shall utilize the Harnack inequalities from Section 2 and a lower bound of the heat kernel in [CY] to prove our Theorems 1, 3 and 5.

Proof of Theorem 1. By the variational principle, it suffices to show that the exists a constant $C_1 > 0$ such that

$$\int_M |\nabla f|^2 \geq \frac{R^m}{R_0^{m+2}} \exp(-C_1(1 + R_0\sqrt{K})) \inf_{a \in \mathbb{R}} \int_M (f - a)^2$$

for any smooth function f defined on M. Let $H(x, y, t)$ be the Neumann heat kernel defined on M. The function

$$F(x, t) = \int_M H(x, y, t) \, f(y) dy$$

solves the heat equation

$$(\Delta - \frac{\partial}{\partial t})F(x, t) = 0$$

on M with Neumann boundary condition

$$\frac{\partial F}{\partial \nu} = 0$$

on ∂M and initial condition $F(x, 0) = f(x)$. Let us now consider the function

$$g(x, t) = \int_M H(x, y, t) \left(f(y) - F(x, t)\right)^2 dy.$$

Clearly,

$$\begin{aligned}
\int_M g(x, t) \, dx &= \int_M \int_M H(x, y, t) \, f^2(y) \, dy \, dx - \int_M F^2(x, t) \, dx \\
&= \int_M f^2(y) \, dy - \int_M F^2(x, t) \, dx \\
&= -\int_0^t \frac{\partial}{\partial s} \int_M F^2(x, s) \, dx \\
&= -2 \int_0^t \int_M F(x, s) \, \Delta F(x, s) \, dx \\
&= 2 \int_0^t \int_M |\nabla F|^2(x, s) \, dx.
\end{aligned}$$

However, if we consider

$$\begin{aligned}
\frac{\partial}{\partial t} \int_M |\nabla F|^2(x, t) \, dx &= 2 \int_M \langle \nabla F, \nabla F_t \rangle(x, t) \, dx \\
&= -2 \int_M \langle \Delta F, F_t \rangle(x, t) \, dx \\
&= -2 \int_M F_t^2(x, t) \, dx,
\end{aligned}$$

we conclude that

$$\int_M |\nabla F|^2(x, t) \, dx \leq \int_M |\nabla f|^2(x) \, dx.$$

Hence, we have the estimate

$$(23) \qquad \int_M g(x,t)\, dx \le 2t \int_M |\nabla f|^2(x)\, dx.$$

On the other hand, since $\left(f(y) - F(x,t) \right)^2 \ge 0$ the function g is also nonnegative. By the definition of R and that $B_p(\frac{R}{2}) \subset M$, we have

$$
\begin{aligned}
\int_M g(x,t)\, dx &\ge \int_{B_p(\frac{R}{2})} g(x,t)\, dx \\
&= \int_{B_p(\frac{R}{2})} \int_M H(x,y,t) \left(f(y) - F(x,t) \right)^2 dy\, dx \\
&\ge \int_{B_p(\frac{R}{2})} \inf_{y \in M} H(x,y,t) \int_M \left(f(y) - F(x,t) \right)^2 dy\, dx \\
&\ge \inf_{a \in \mathbb{R}} \int_M \left(f(y) - a \right)^2 dy \int_{B_p(\frac{R}{2})} \inf_{y \in M} H(x,y,t)\, dx.
\end{aligned}
$$

(24)

Using (23) and (24), we conclude that

$$\eta_1 \ge (2t)^{-1} \int_{B_p(\frac{R}{2})} \inf_{y \in M} H(x,y,t)\, dx$$

for all $t > 0$. Note that by Lemma 1, the Harnack inequality of Li-Yau, for any $x, z \in B_p(\frac{R}{2})$ we have

$$H(x,y,t) \ge H\left(z,y,\frac{t}{2}\right) C_4 \exp\left(-C_8 \left(\frac{R^2}{t} + Kt + \frac{t}{R^2} \right) \right)$$

for some constants C_4, $C_5 > 0$ depending only on m. Hence

$$
\begin{aligned}
&\int_{B_p(\frac{R}{2})} \inf_{y \in M} H(x,y,t)\, dx \\
&\ge V_p\left(\frac{R}{2}\right) \inf_{x \in B_p(\frac{R}{2}), y \in M} H(x,y,t) \\
&\ge C_4 \exp\left(-C_5 \left(\frac{R^2}{t} + Kt + \frac{t}{R^2} \right) \right) \inf_{y \in M} \int_{B_p(\frac{R}{2})} H\left(z,y,\frac{t}{2}\right) dz,
\end{aligned}
$$

and

$$(25) \quad \eta_1 \ge \frac{C_4}{t} \exp\left(-C_5 \left(\frac{R^2}{t} + Kt + \frac{t}{R^2} \right) \right) \inf_{y \in M} \int_{B_p(\frac{R}{2})} H\left(z,y,\frac{t}{2}\right) dz.$$

To estimate the right hand side, we use an argument of Li-Yau [LY2] together with the Cheeger-Yau [CY] comparison theorem. Let $\phi(x) = \phi(r(x))$ be a function of the distance r to the center point p. We choose $0 \le \phi \le 1$ to have the properties that

$$\phi = 1 \qquad \text{on } B_p\left(\frac{R}{2} - \delta\right),$$

$$\phi = 0 \qquad \text{on } B_p\left(\frac{R}{2}\right) \setminus B_p\left(\frac{R}{2} - \delta\right),$$

and

$$\phi' = \frac{\partial \phi}{\partial r} \le 0.$$

Clearly, the solution to the heat equation

$$\Phi(x,t) = \int_M H(z,x,t)\,\phi(z)\,dz$$

satisfies

$$\Phi(x,t) \leq \int_{B_p(\frac{R}{2})} H(z,x,t)\,dz.$$

Let $B(K,\frac{R}{2})$ and $B(K,R_0)$ be the geodesic balls of radius $\frac{R}{2}$ and R_0 with a fixed center $\bar{p}$ on the simply connected model $\bar{M}$ of constant $-K$ sectional curvature. If $\bar{H}(\bar{x},\bar{y},t)$ denotes the Neumann heat kernel on $B(K,R_0)$, and $\bar{r}$ denotes the distance to the point $\bar{p}$, then we can define the corresponding solution of the Neumann heat equation by

$$\bar{\Phi}(\bar{x},t) = \int_{B(K,R_0)} \bar{H}(\bar{z},\bar{x},t)\,\phi(\bar{r}(\bar{z}))\,d\bar{z}.$$

By the uniqueness of the heat equation and the fact that ϕ is rotationally symmetric, $\bar{\Phi}$ must also be rotationally symmetric. Hence, we can write $\bar{\Phi}(\bar{y},t) = \bar{\Phi}(\bar{r}(\bar{y}),t)$. In particular, $\bar{\Phi}'(0,t) = \frac{\partial\bar{\Phi}}{\partial\bar{r}}(0,t) = 0$. Also, by the Neumann boundary condition, $\bar{\Phi}'(R_0,t) = \frac{\partial\bar{\Phi}}{\partial\bar{r}}(R_0,t) = 0$. Using the same argument as in [LY2], we can show that $\bar{\Phi}'(\bar{r},t) \leq 0$ for all $0 \leq \bar{r} \leq R_0$ and for all $t \geq 0$. Hence the Cheeger-Yau comparison theorem implies that the transplanted function define by $\bar{\Phi}(y,t) = \bar{\Phi}(r(y),t)$ is a subsolution,

$$\left(\Delta - \frac{\partial}{\partial t}\right)\bar{\Phi}(y,t) \geq 0$$

in the sense of distribution on M, with initial condition

$$\bar{\Phi}(y,0) = \phi(y).$$

The difference $\Psi = \Phi - \bar{\Phi}$ is therefore a supersolution of the heat equation of $M \times [0,\infty)$ which vanishes on $M \times \{0\}$. We also claim that the normal derivative $\frac{\partial\Psi}{\partial\nu}$ with respect to the outward normal must be non-negative on ∂M. Clearly, by the boundary condition of Φ, it suffices to check that

$$\frac{\partial\bar{\Phi}}{\partial\nu} \leq 0.$$

Indeed, by the fact that $\bar{\Phi}$ is rotationally symmetric, for any $y \in \partial M$, we have

$$\frac{\partial\bar{\Phi}}{\partial\nu} = \bar{\Phi}'(r(y))\,\langle\frac{\partial}{\partial r},\nu\rangle(y).$$

The assertion follows from the fact that $\bar{\Phi}' \leq 0$, and the fact that $\langle\frac{\partial}{\partial r},\nu\rangle \geq 0$ on ∂M because M is geodesically star-shaped with respect to p.

The parabolic maximum principle implies that Ψ must have its minimum occurred on $(M \times \{0\}) \cup (\partial M \times [0,\infty))$. If the minimum of Ψ occurred on $\partial M \times (0,\infty)$, then the Hopf boundary lemma implies that $\frac{\partial\Psi}{\partial\nu} < 0$ violating the Neumann boundary condition. Hence, the minimum of Ψ must be achieved on $M \times \{0\}$, which

has value 0. This implies that $\Phi \geq \bar{\Phi}$. In particular, for $y \in M$,

$$\int_{B_p(\frac{R}{2})} H(z, y, \frac{t}{2}) \, dz \geq \Phi(y, \frac{t}{2})$$

(26)
$$\geq \bar{\Phi}(r(y), \frac{t}{2})$$

$$\geq \overline{\Phi}(R_0, \frac{t}{2}).$$

Applying Lemma 2, the Harnack inequality from [LY2], on the function $\bar{\Phi}$, we have

(27) $$\overline{\Phi}(R_0, \frac{t}{2}) \geq C_6 \, \bar{\Phi}(0, \frac{t}{4}) \exp\left(-C_7(\frac{R_0^2}{t} + Kt)\right)$$

for some constants C_6, $C_7 > 0$ depending only on m. Since $\bar{\Phi}' \leq 0$, we conclude that $\bar{\Phi}(0, \frac{t}{4}) \geq \bar{\Phi}(\bar{r}, \frac{t}{4})$ for all $0 \leq \bar{r} \leq R$. This together with the fact that $\bar{\Phi}$ satisfies the heat equation

$$\left(\bar{\Delta} - \frac{\partial}{\partial t}\right) \bar{\Phi}(\bar{r}, t) = 0$$

implies that $\frac{\partial \bar{\Phi}}{\partial t}(0, t) \leq 0$ for all $t > 0$. In particular,

$$\bar{\Phi}(0, \frac{t}{4}) \quad \geq \quad \lim_{t \to \infty} \bar{\Phi}(0, t)$$

$$= \quad \int_{B(K, R_0)} \frac{\phi(\bar{r}(\bar{z}))}{V(K, R_0)} \, d\bar{z}$$

where $V(K, R_0)$ is the volume of $B(K, R_0)$. Taking the limit $\delta \to 0$, we conclude that

$$\bar{\Phi}(0, \frac{t}{4}) \geq \frac{V(K, \frac{R}{2})}{V(K, R_0)}.$$

Combining with (25), (26) and (27), we have

$$\eta_1 \geq C_{21} (t)^{-1} \exp\left(-C_{22}(\frac{R^2}{t} + \frac{t}{R^2} + \frac{R_0^2}{t} + Kt)\right) \frac{V(K, \frac{R}{2})}{V(K, R_0)},$$

for some constants C_{21}, $C_{22} > 0$. Setting

$$t = \frac{R_0^2}{1 + R_0\sqrt{K}}$$

and using the estimates

$$t^{-1} \geq R_0^{-2}$$

and

$$\frac{R_0^2 K}{1 + R_0\sqrt{K}} \leq 1 + R_0\sqrt{K},$$

we conclude that

$$\eta_1 \geq R_0^{-2} \exp(-C_{23}(1 + R_0\sqrt{K})) \frac{V(K, \frac{R}{2})}{V(K, R_0)}$$

for some constant $C_{23} > 0$ depending only on m. The theorem now follows by observing that there exists constants $C_{24} > 0$ and $C_{25} > 0$ depending only on m

such that

$$\bar{r}^m \exp\left(C_{24}(1 + \bar{r}\sqrt{K})\right) \leq V(K, \bar{r})$$

(28)
$$\leq \bar{r}^m \exp\left(C_{25}(1 + \bar{r}\sqrt{K})\right).$$

This completes the proof of Theorem 1. □

It is proved in [G] that if a manifold satisfies volume doubling property and it has a positive Neumann eigenvalue, then one can use a covering argument to give a lower estimate of the first Dirichlet eigenvalues for the proper sub domains in terms of the volume doubling constant and the first Neumann eigenvalue. Now, we can apply a result in [G] to prove our Theorem 2.

Proof of Theorem 2. We note that the volume doubling property implied by (28) asserts that for any $B_x(2r) \subset B_p(R)$, we have

(29)
$$\frac{V(B_x(2r))}{V(B_x(r))} \leq \frac{V(K, 2r)}{V(K, r)}$$
$$\leq C_v(R),$$

for some positive constant $C_v(R)$ depending on $m, R\sqrt{K}$. Also, Theorem 1 implies that, for any $B_x(r) \subset B_p(R)$,

$$\int_{B_x(r)} |\nabla f|^2 \geq \frac{1}{R^2} \exp(-C_1(1 + R\sqrt{K})) \inf_{a \in \mathbb{R}} \int_{B_x(r)} (f - a)^2.$$

Therefore, we can apply Theorem 1.4 of [G] to obtain a lower bound for the first Dirichlet eigenvalue.

Next, we shall modify the proof of Theorem 1 and use the Harnack inequality Corollary 4 from Section 2 to give a proof of Theorem 3. Let u_1, u_2 be the first two eigenfunctions with eigenvalues λ_1, λ_2 respectively for the problem (2). By setting $\Gamma = \lambda_2 - \lambda_1$, $w = \frac{u_2}{u_1}$ and $h = \log u_1$, we get that

$$\Delta w + 2\langle \nabla h, \nabla w \rangle = -\Gamma w \qquad \text{in } M,$$
$$\frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \partial M.$$

We consider a new metric ds_1^2 defined by

$$ds_1^2 = u_1^{\frac{4}{m-2}} ds^2$$

with the volume element $dv_1 = u_1^{\frac{2m}{m-2}} dv$. In terms of this metric, the equation is equivalent to

$$u_1^{\frac{4}{m-2}} \Delta_1 w = -\Gamma w \qquad \text{in } M,$$
$$\frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \partial M,$$

where Δ_1 denotes the Laplacian operator with respect to the metric ds_1^2. □

Proof of Theorem 3. To prove the theorem, by the variational principle, it suffices to prove there exists a constant $\alpha_2 > 0$ such that

$$\int_M |\nabla_1 f|^2 \geq \frac{R^m}{R_0^{m+2}} e^{-\alpha_2} \inf_{a \in \mathbb{R}} \int_M (f - a)^2 u_1^{-\frac{4}{m-2}},$$

where we are integrating with respect to the new metric ds_1^2. If $W(x, y, t)$ be the Neumann fundamental solution of the problem

$$\left(u_1^{\frac{4}{m-2}}\Delta_1 - \frac{\partial}{\partial t}\right)u(x, t) = 0 \qquad \text{in } M,$$

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial M.$$

The function

$$F(x, t) = \int_M W(x, y, t) \, f(y) \, u_1^{-\frac{4}{m-2}}(y) \, dv_1(y)$$

solves the equation

$$(u_1^{\frac{4}{m-2}}\Delta_1 - \frac{\partial}{\partial t})F(x, t) = 0$$

in M with the initial condition $F(x, 0) = f(x)$. Let us now consider the function

$$g(x, t) = \int_M W(x, y, t) \, (f(y) - F(x, t))^2 \, u_1^{-\frac{4}{m-2}}(y) \, dv_1(y),$$

and follow a similar calculations as in the proof of Theorem 1, we obtain that

$$\int_M g(x, t) \, u_1^{-\frac{4}{m-2}}(x) \, dv_1(x) = 2 \int_0^t \int_M |\nabla_1 F(x, s)|^2 \, dv_1(x) \, ds.$$

and

$$\int_M |\nabla_1 F(x, t)|^2 \, dv_1(x) \le \int_M |\nabla_1 f(x)|^2 \, dv_1(x).$$

Therefore, we have

$$\int_M g(x, t) \, u_1^{-\frac{4}{m-2}}(x) \, dv_1(x) \le 2t \int_M |\nabla_1 f|^2(x) \, dv_1(x).$$

On the other hand, we see that

$$\int_M g(x, t) \, u_1^{-\frac{4}{m-2}}(x) \, dv_1(x)$$

$$\ge \int_{B_p(\frac{R}{2})} \inf_{y \in M} W(x, y, t) \, u_1^{-\frac{4}{m-2}}(x)$$

$$\times \int_M (f(y) - F(x, t))^2 \, u_1^{-\frac{4}{n-2}}(y) \, dv_1(y) \, dv_1(x)$$

$$\ge \inf_{a \in \mathbb{R}} \int_M (f(y) - a)^2 \, u_1^{-\frac{4}{m-2}}(y) \, dv_1(y)$$

$$\cdot \int_{B_p(\frac{R}{2})} \inf_{y \in M} W(x, y, t) \, u_1^{-\frac{4}{m-2}}(x) \, dv_1(x).$$

In the second integral, we can write it in terms of the metric ds^2, and use the Corollary 4 in Section 2 to get a lower estimate on

$$\inf_{\substack{x \in B_p(\frac{R}{2}) \\ y \in M}} \frac{u_1(x)}{u_1(y)}.$$

Thus, we have

$$\int_{B_p(\frac{R}{2})} \inf_{y\in M} W(x,y,t)\, u_1^{-\frac{4}{m-2}}(x)\, dv_1(x)$$

$$\geq\ C_h\, e^{\lambda_1 t}\int_{B_p(\frac{R}{2})} \inf_{y\in M} G(x,y,t)\, dv(x)$$

$$\geq\ C_h\, e^{t\inf q}\int_{B_p(\frac{R}{2})} \inf_{y\in M} G(x,y,t)\, dv(x),$$

where $G(x,y,t)$ is the Neumann fundamental solution of the equation

$$\Big(\Delta - q - \frac{\partial}{\partial t}\Big) u(x,t) = 0.$$

Note that the $G(x,y,t)$ is bounded from below by the Neumann fundamental solution $e^{-t\sup q}\, H(x,y,t)$ of the equation

$$\Big(\Delta - \sup q - \frac{\partial}{\partial t}\Big) u(x,t) = 0$$

on $M\times(0,T)$, where $H(x,y,t)$ is the Neumann heat kernel on M. We obtain that

$$\int_M g(x,t)\, u_1^{-\frac{4}{m-2}}(x)\, dv_1(x) \geq C_h\, e^{t\,\omega(q)}\int_{B_p(\frac{R}{2})} \inf_{y\in M} H(x,y,t)\, dv(x).$$

The proof of Theorem 1 implies that there exist positive constants C_{26} and C_{27} depending only on m such that

$$\int_M |\nabla_1 f|^2 \geq \frac{C_{26}\, V(K,\frac{R}{2})\, C_h\, \exp\Big(-C_{27}(\frac{R^2}{t} + \frac{R_0^2}{t} + (\omega(q)+K+\frac{1}{R^2})t)\Big)}{2t\, V(K,R_0)}$$

$$\cdot \inf_{a\in\mathbb{R}} \int_M (f-a)^2\, u_1^{-\frac{4}{m-2}}.$$

This implies that

$$\Gamma\ \geq\ \frac{C_{27}\, V(K,\frac{R}{2})\, C_h\, \exp\Big(-C_{28}(\frac{R^2}{t} + \frac{R_0^2}{t} + (\omega(q)+K+\frac{1}{R^2})t)\Big)}{2t\, V(K,R_0)}$$

We complete the proof by setting

$$t\ =\ \frac{R_0^2}{1 + R_0\sqrt{K+\omega(q)}},$$

and using the fact that C_h depends on m, $R_0\sqrt{K}$ and $\omega(q)R_0^2 + 1$. $\qquad\square$

Following the same argument as in the proof of Theorem 2, we shall justify the volume doubling property and the existence of Poincaré inequality with respect to the conformal metric defined in the proof of Theorem 3.

Proof of Theorem 4. Since u_1 satisfies the Harnack inequality and its L^2-norm is 1, we have upper and lower bound for u_1 as follows

$$\frac{C_h}{\sqrt{V(B_p(R))}} \leq u_1(x) \leq \frac{C_h^{-1}}{\sqrt{V(B_p(R))}}.$$

Therefore, with respect to the metric ds_1^2, for any $B_x(2r) \subset B_p(R)$,

$$\frac{V(B_x(2r))}{V(B_x(r))} \leq \alpha_{11},$$

for some constant α_{11} depends on m, $R\sqrt{K}$ and $\omega(q)R^2+1$. Also, by using Theorem 3, we have

$$\int_{B_x(r)} |\nabla_1 f|^2 \geq \frac{\alpha_{12}}{R^2} \exp(-\alpha_2) \inf_{a \in \mathbb{R}} \int_{B_x(r)} (f-a)^2,$$

where $\alpha_{12} = \left(\frac{\sqrt{V(B_p(R))}}{C_h}\right)^{\frac{4}{m-2}}$. Then, we can apply the argument as in the proof of Theorem 1.4 in [G] to get our bound. $\qquad\square$

Proof of Theorem 5. It is obvious that if we follow the proof of Theorem 3 and use the Harnack inequality, Corollary 5 from Section 2, then we can obtain our lower bound for the gap. $\qquad\square$

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